Fair Division of Objects with Money

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a follow up to the paper

Guarantees in Fair Division: general or monotone preferences

with Anna Bogomolnaia and Richard Stong

https://www.gla.ac.uk/media/Media_700119_smxx.pdf

- the manna: *indivisible* objects: mixed goods and bads, *all must go*

- money is available for compensations

Dissolution of partnership: divorce, bankruptcy with both assets and liabilities. Inheritance: heirlooms, art, jewelry, real estate, financial assets;

jobs between contractors (profitable or not); rooms and rent between flatmates (active on *SPLIDDIT.org*)

we look for rules to allocate the indivisible objects that are easy to play and "demonstrably fair" two familiar rules:

- *Divide and Choose* **(no cash)**: each player is *guaranteed* “his good half”

- *Texas Shoot Out* **(with cash)**: each player is *guaranteed* at least half of her value for the venture

common features: simple messages requiring small cognitive effort; reveal little: preserve privacy

worst case implementation of fairness: *no matter what the other parties do;* strategize at your own risk, *only an equal cut is safe*

Texas Shoot Out with n agents

- auction the bundled manna A
- split the winning bid equally between all agents (including the winner)
- $\Longrightarrow$ each agent is guaranteed $\frac{1}{n}u(A)$ by reporting truthfully

assigning the whole manna "is the efficient way" if utilities are superaddive (and symmetric)

but de-bundling the manna is more efficient if utilities are subadditive

Divide and Choose can do the right thing in both cases

but its generalization to n agents requires some work

and it treats the Divider and Choosers differently *what we do*: discuss two types of division rules using cash transfers to smooth out indivisibilities (as in TSO), *and* where the agents can implement any partition of the objects (as in D&C)

- the fair version of **Divide and Choose** for n agents

- the new family of **Bid and Choose** rules, more privacy-preserving than D&C; and ask the manager to select a benchmark price of objects the model

A a finite set of objects; N the set of n agents

to do: pick a partition $\Pi = \{S_i\}_{i=1}^n$ and cash transfers t_i such that $\sum_N t_i = 0$

utilities are *quasi-linear*: $u_i(S_i) + t_i$; status quo ante $u_i(\varnothing) = 0$

utilities are *combinatorial* $u_i \in \mathbb{R}^{2^A}$

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an efficient allocation of the objects maximises the sum $\sum_N u_i(S_i)$ over all partitions, irrespective of cash transfers: this is computationally and cognitively difficult

our simple rules, like Divide and Choose and the Texas Shoot Out, do not deliver fully efficient allocations; they do not require agents to form, let alone report, detailed utilities the mixed-monotone domain $\mathcal{M}$: each object is either *good*: $\partial_a u(S) = u(S \cup \{a\}) - u(S) \geq 0$ for all $S \subset A$, or *bad:* $\partial_a u(S) \leq 0$ for all $S \subset A$. Write $A = A^+ \cup A^-$

the subadditive domain $\mathcal{S}ub$: $u(S \cup T) \leq u(S) + u(T)$ if $S \cap T = \varnothing$ the whole is less than the parts, e.g. decreasing marginal utility

the superadditive domain $\mathcal{S}up$: $u(S \cup T) \geq u(S) + u(T)$ if $S \cap T = \varnothing$ e. g. decreasing marginal cost of tasks

most general: the full domain $\mathcal{D} = \mathbb{R}^{2^A}$ allows more complicated externalities (as in team selection) a n-partition of A is $\Pi = \{S_k\}_{k=1}^n$; *shares can be empty*

a compensated n-partition is $\Delta = (\Pi, \Lambda) = \{(S_k, \lambda_k)\}_{k=1}^n$ where the cash transfers are balanced: $\sum_1^n \lambda_k = 0$

a simple Lemma: fix for each agent $j \in N$ a balanced vector of transfers $\Lambda^j = (\lambda_k^j)_{k=1}^n$. Then there exists an assignment $\{1, \cdots, n\} \ni k \to \sigma(k) = j \in N$ with a non negative surplus:

$$\sum_1^n \lambda_k^{\sigma(k)} \leq 0$$

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k	1	2	3	4	$\sum_k \lambda_k$
λ^1	0	-3^*	4	-1	0
λ^2	-2	1	1	0^*	0
λ^3	-4	6	-1^*	-1	0
λ^4	-5^*	1	3	1	0

D&C$_n$ *Divide and Choose with cash among n agents*
1 Divider, $n-1$ Choosers

Divider picks a compensated partition $(\Pi, \Lambda^D) = \{(S_k, \lambda_k^D)\}_{k=1}^n$ then each Chooser C_j, $j \in N \diagdown \{D\}$, picks balanced transfers $\Lambda^j = (\lambda_k^j)_{k=1}^n$.

the rule picks an assignment σ with the largest surplus (the smallest sum of transfers), assigns $(S_k, \lambda_k^{\sigma(k)})$ to agent $\sigma(k)$ and distributes equally the surplus to achieve balanced transfers.

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Π	S_1	S_2	S_3	S_4	$\sum_k \lambda_k$
λ^1	0	-3^*	4	-1	0
λ^2	-2	1	1^*	0	0
λ^3	-4^*	4	0	0	0
λ^4	-3	1	3	-1^*	0

surplus: -7

what Guarantees are implemented by D&C$_n$?

the Divider exploits optimally the non-additivity of u:

$$Maxmin(u;n) = \max_{\mathcal{P}_n} \min_k \{u(S_k) + t_k\} = \frac{1}{n} \max_{\mathcal{P}_n} \sum_1^n u(S_k)$$

($\mathcal{P}_n$ is the set of partitions)

a Chooser's Guarantee:

$$minMax(u;n) = \min_{\mathcal{P}_n} \max_k \{u(S_k) + t_k\} = \frac{1}{n} \min_{\mathcal{P}_n} \sum_1^n u(S_k)$$

Fair Share Guarantee: the utility level $\Gamma(u; n; A)$ an agent with utility u can secure, through some division rule; the agent has no clue about utilities and actions of other agents

built-in Anonymity

feasibility needs to be checked!

simplest feasible Guarantee: $\frac{1}{n}u(A)$

two general facts in **non atomic** problems of fair division

$Maxmin(u;n)$ *is the best possible guarantee*, in any mechanism (by the unanimity argument): Γ is feasible $\Longrightarrow \Gamma(u;n;A) \leq Maxmin(u;n)$

$minMax(u;n)$ is (weakly) lower than $Maxmin(u;n)$, and is *always* a feasible Guarantee

(main Theorem in BMS paper)

(recall from Buddish (2011) that for **atomic** problems with additive utilities and no money, $Maxmin \leq minMax$)

in some small domains where utilities are "similar", $Maxmin$ is a feasible Guarantee, *hence the best ever*

examples

utilities are superadditive $\Longrightarrow Maxmin(u_i; n) = \frac{1}{n}u_i(A)$ is feasible

utilities are subadditive and $n \geq |A| \Longrightarrow Maxmin(u; n) = \frac{1}{n}\sum_a u(a)$ is feasible but this breaks down when utilities are more diverse: for instance some subadditive, some superadditive

Example 1: three identical goods, two agents, one superadditive, one subadditive

	$\varnothing$	1 good	2 goods	3 goods			$Maxmin$	$minMax$
$\mathcal{S}up \ni u$	0	2	8	18	$\Longrightarrow$	u	9	5
$\mathcal{S}ub \ni v$	0	10	14	16		v	12	8

maximal surplus 18 $\Longrightarrow (Maxmin(u), Maxmin(v))$ is not feasible

Example 2: four goods a, b, c, d, two agents, utilities are neither sub- nor super-additive

$$\begin{array}{ccccc} & ab & cd & ac & bd \\ u & 10 & 10 & 0 & 0 \\ v & 0 & 0 & 2 & 10 \end{array} \Longrightarrow \begin{array}{ccc} & Maxmin & minMax \\ u & 10 & 0 \\ v & 6 & 0 \end{array}$$

maximal surplus 10 $\Longrightarrow (Maxmin(u), Maxmin(v))$ is not feasible;

large “duality gap”! Divide and Choose is very unfair we can remove the Divider's advantage in D&C$_n$ by auctioning the role of Divider (old trick: Crawford 1979, Demange 1984)

this Bid-Divide-and-Choose rule is anonymous and implements the Guarantee

$$\Gamma^{BDC}(u;n) = \frac{1}{n} Maxmin(u;n) + \frac{n-1}{n} minMax(u;n)$$

we propose a "better" family of division rules the family of *Bid and Choose rules*

a rule is defined by a price vector $p \in \mathbb{R}^A$ for the objects (perhaps a market price estimate): $p_S = \sum_{a \in S} p_a$ is the cost of the share S

Definition of B&C$_n^p$: there are $n-1$ rounds of bidding

Round 1: each agent i bids λ_i; the highest bidder, say i^* wins: she can pick any share S_{i*} in A (including $\varnothing$ and A) but it costs her $p_{S_{i*}}$ on top of her bid λ_{i*}. Her net payment is $t_{i*} = \lambda_{i*} + p_{S_{i*}}$: each agent $i \neq i^*$ receives $\frac{1}{n-1} t_{i*}$

if $S_{i*} = A$ Stop; otherwise go to

Round 2: repeat among $N \diagdown i^*$ and $A \diagdown S_{i*}$

the loser in the last round gets the remaining share at no cost if $p \equiv 0$ and all objects are "good" the $\text{B\&C}_n^0$ rule is simply TSO_n

$p_S \geq 0$ means that S is “objectively valuable”, while $p_S \leq 0$ means that clearing S is “objectively a favour”, and the rule compensates accordingly

a related approach: competitive Fair Division moderated by market prices (d’All Aglio (2020))

Theorem

$i)$ assume $u \in \mathcal{M}$ (each object is a good or a bad) and/or $n = 2$; the rule B&C$_n^p$ guarantees

$$\Gamma^p(u;n) = \frac{1}{n}[(u-p)^{\max} + (u-p)^{\min} + p_A]$$

(this is a lower bound in the general domain $\mathcal{D} = \mathbb{R}^{2^A}$)

interpretation: $\Gamma^p(u;n)$ is a fair share of the exogeneous value of manna, corrected by my largest over- and under-valuations proof of property $i)$ for $n = 2$

my bid λ guarantees

$$\min\{(u-p)^{\max} - \lambda, \min_{\varnothing \subseteq T \subset A} u(T^c) + (\lambda + p_T)\}$$

$$\Longrightarrow \lambda^* = \frac{1}{2}\{(u-p)^{\max} - \min_{\varnothing \subseteq T \subset A}(u(T^c) - p_{T^c}) - p_A\}$$

induction to $n = 3$: choose the bid λ to maximize

$$\min\{((u-p)^{\max} - \lambda), \min_T \{\frac{1}{2}(p_T + \lambda) + \frac{1}{2}[(u-p)^{\max}_{T^c} + (u-p)^{\min}_{T^c} + p_{T^c}]\}\}$$

$ii)$ *for each* $u \in \mathbb{R}^{2^A}$ *and* n*:* $minMax(u;n) \leq \Gamma^p(u;n) \leq Maxmin(u;n)$

$iii)$ *the profile of Guarantees* Γ^p *is Second Best:* $\sum_1^n \Gamma(u_i;n) \geq \min_{\mathcal{P}_n} \sum_1^n u_i(S_i)$ *for all* $u \in \mathbb{R}^{2^A \times N}$

comparison with BDC_n^p:

as n grows $\Gamma^{BDC}(u;n)$ gets very close to $minMax(u;n)$

$\Gamma^{BDC}(u;n)$ also fails the "Second Best" property role of the design parameter p: the Guarantee Γ^p favors subadditive and penalises superadditive utilities

Theorem

for any $p \in \mathbb{R}^A$

$$\Gamma^p(u;n) \geq \frac{1}{n}u(A) \text{ if } u \in \mathcal{S}ub \text{ ; } \Gamma^p(u;n) \leq \frac{1}{n}u(A) \text{ if } u \in \mathcal{S}up$$

what empirical choice of p ? *for any* $u \in \mathcal{D}$ *and* n, $\Gamma^p(u;n) = \frac{1}{n}u(A)$ *if* p *is either large enough positive or small enough negative*

a clue when objects are identical

notation: $u(S) = U_{|S|}$ and $|A| = \alpha$

$$\Gamma^p(u;n) = \frac{1}{n}[\max_{1\le k\le\alpha}(U_k - kp) + \min_{1\le k\le\alpha}(U_k - kp) + \alpha p]$$

Proposition:

if $k \longrightarrow U_k$ *is convex the minimum of* $\mathbb{R} \ni p \longrightarrow \Gamma^p(U;n)$ *is for* $p_a = \frac{U\alpha}{\alpha}$ *(for all* a*)*

if $k \longrightarrow U_k$ *is concave the maximum of* $\mathbb{R} \ni p \longrightarrow \Gamma^p(U;n)$ *is for* $p_a = \frac{U\alpha}{\alpha}$ *(for all* a*)*

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Example 1: two agents, three goods total surplus is 18

	$\varnothing$	1 good	2 goods	3 goods
$\mathcal{S}up \ni u$	0	2	8	18
$\mathcal{S}ub \ni v$	0	10	14	16

	$Maxmin$	$minMax$	Γ^{BDA}	Γ^{0}	$\Gamma^{5.7}$
u	9	5	7	9	7.2
v	12	8	10	8	10.2

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Example 2: four goods a, b, c, d, two agents, utilities are neither sub- nor super-additive

	ab	cd	ac	bd			$Maxmin$	$minMax$	Γ^{BDA}	Γ^0	$\Gamma^{2.5}$
u	10	10	0	0	$\Longrightarrow$	u	10	0	5	5	5
v	0	0	2	10		v	6	0	3	5	4.75

difficult mathematical questions

the most desirable Guarantees Γ are **Maximal**

for all feasible n-person $\widetilde{\Gamma} : \{\Gamma(u;n) \leq \widetilde{\Gamma}(u;n)$ *for all* $u \in \mathcal{D}\} \Longrightarrow \Gamma = \widetilde{\Gamma}$

- *a fixed n-partition* Π *gives the Guarantee* $\Gamma^{\Pi}(u;n) = \frac{1}{n}\sum_N u(S_j)$*; it is maximal (on $\mathcal{D}$ and $\mathcal{M}$) if and only if* $|\Pi| = 1$ *or n; what about convex combinations of several such* Γ^{Π} *?*

- *for $n = 2$ both Guarantees* $\Gamma^{p}(\cdot;2)$ *and* $\Gamma^{BDA}(\cdot;2) = \frac{1}{2}(Maxmin + minMax)$ *are maximal (on $\mathcal{D}$ and $\mathcal{M}$);* **but not for** $n \geq 3$*: how to improve them ?*

our model, though very common in real life, is not often discussed by the fair division literature

the classic cake-cutting model (Steinhaus 1948) rules out indivisible objects

recent work at the interface of Computer Science and Economics allocates indivisible objects

- either with randomisation in lieu of cash transfers: Bogomolnaia and Moulin (2001), Buddish (2011)

- or without lotteries or cash, and looks for “approximate fairness” (Procaccia and Wang (2014), Caragianis et al. (2016))

take home points:

- Divide and Choose is easily adapted to n agents problems, but does not offer good Guarantees

- the Bid and Choose rules are equally easy to implement once an objective benchmark value can be agreed upon; this needs to be tested experimentally

- their efficiency benefit above the Texas Shoot Out and Divide and Choose needs to be evaluated numerically

Thank You